Counter-intuitive results from the field of the Hyper-reals

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Introduction

- Inspired from the paper *Infinitesimals in Modern Mathematics* presented in last Fall's Seaway meeting
- Some repeat information, some new information
- The Hyperreals ***R** is an enlargement of **R**
- ***R** contains both infinite and infinitesimal values
- Very similar and very dissimilar properties
- Some find proofs of ***R** more intuitive than those of **R**
- Paper & slides available to view and download at

  http://www.jonhoyle.com/MAASeaway

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Outline

- Construct $^*\mathbf{R}$ out of $\mathbf{R}^\infty$
- Add a Hyperreal Equivalence Relation
- Look at interesting properties:
  - Infinities & infinitesimals
  - Halos and Unique Shadows
  - The Transfer Principle
  - Countability & Hyper-Countability
  - Dedekind Incompleteness
  - Nonstandard Proofs
- Conclusion
- Q & A

Construction of *R

- Begin with $\mathbf{R}^\infty$, the set of ordered sequences of $\mathbf{R}$:

$$< 0, 1, 0, 1, \ldots >$$

$$< 2, 3, 5, 7, 11, \ldots >$$

$$< -1, \pi, 0.0001, 10^{10}, \sqrt{17}, \ldots >$$

- Identify the reals as a subset, eg: 3 = $< 3, 3, 3, \ldots >$
- Define arithmetic and extended functions:

$$a + b = < a_0 + b_0, a_1 + b_1, a_2 + b_2, \ldots >$$

$$a \times b = < a_0 \times b_0, a_1 \times b_1, a_2 \times b_2, \ldots >$$

$$a \div b = < a_0 \div b_0, a_1 \div b_1, a_2 \div b_2, \ldots >$$

$$a^b = < a_0^{b_0}, a_1^{b_1}, a_2^{b_2}, a_3^{b_3}, \ldots >$$

$$f(a) = < f(a_0), f(a_1), f(a_2), f(a_3), \ldots >$$

Hyperreal Equivalence Relation

- Divide all subsets of **N** into *"large"* and *"small"*:
  - All finite subsets of N being *small*
  - All cofinite subsets of N being *large*
  - Complement of a *large* set is *small*, and vice-versa
- $< a_0, a_1, \ldots > = < b_0, b_1, \ldots >$ holds when the agreement set $< a_0 = b_0, a_1 = b_1, \ldots >$ is large
- Using a *non-principal ultrafilter* on **N**, we can define an equivalence relation satisfying our *large* and *small*
- $^*\mathbf{R}$ is the set of equivalence classes over $\mathbf{R}^\infty$
- $^*\mathbf{R}$ is a totally ordered field

Infinities Both Great & Small

- Ordered: ***x < y*** when the set $\{\, i \mid x_i < y_i \,\}$ holds true for a *large* set of indices
- Let $\omega = < 1, 2, 3, \ldots >$
- We see that $\omega > n$, *for all* $n \in \mathbf{N}$
- Thus ω is an infinite element of $^*\mathbf{N}$
- Orders of infinity: $\omega, \omega^2, \omega^\omega, \omega^{\omega^\omega}, \ldots$ *etc.*
- Let $\varepsilon = {}^1/_\omega = < 1, \tfrac{1}{2}, \tfrac{1}{3}, \tfrac{1}{4}, \ldots >$
- We see that $\varepsilon < r$, *for all* $r \in \mathbf{R}^+$
- Thus, ε is an infinitesimal
    - Orders of the infinitely small: $\varepsilon, \varepsilon^2, \varepsilon^\omega, \ldots$ *etc.*

Halos & Unique Shadows

- For $x, y \in {}^*\mathbf{R}$, if $x - y$ is an infinitesimal, we say that x is *infinitely close* to y, written $x \approx y$. Eg: $\omega^{\varepsilon} \approx 1$
- The set of all hyperreals infinitely close to x is called *the halo of* x, denoted $hal(x)$. Eg: $\mathbf{I} = hal(0)$.
- The set of hyperreals a finite distance from x is called *the galaxy of* x, denoted $gal(x)$. Eg: $\mathbf{R} \subseteq gal(0)$.
- Every finite hyperreal x is infinitely close to ***exactly one*** standard real r. r is called *the shadow of* x.
- Any finite hyperreal x can be expressed as $x = r + i$, where $r \in \mathbf{R}$ and i is an infinitesimal:

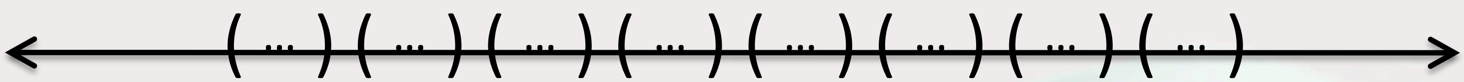
  r is called *the standard part* of x.

  i is called *the nonstandard part* of x.

Topological View

- We think of **R** (very sloppily) as series of points:

- Equally as sloppily, we can think of ***R** as series of non-overlapping open intervals:



The Transfer Principle

- Analytic study of $^*\mathbf{R}$ is called *Nonstandard Analysis*.
- **Transfer Principle**: *First order statements* about $\mathbf{R}$ can be reinterpreted as *first order statements* about $^*\mathbf{R}$.
- This *reinterpretation* involves modifying the statements with the **-transformation*.
- For example, in standard analysis, the true statement:

$$\forall x, \exists n \in \mathbf{N} \ni: n > x$$

is false in NSA. However, it is true when *-transformed:

$$\forall x, \exists n \in {}^*\mathbf{N} \ni: n > x$$

- *-transformed sets of $^*\mathbf{R}$ are called *internal sets*.
    - Remaining sets are called *external sets*.

Countability & Hyper-Countability

- From standard analysis, $|\mathbf{N}| = |\mathbf{Q}| = \aleph_0 < |\mathbf{R}| = \mathfrak{c}$.
- From Transfer, we might therefore expect $|{}^*\mathbf{N}| < |{}^*\mathbf{R}|$.
- However, $|{}^*\mathbf{N}| = |{}^*\mathbf{Q}| = |{}^*\mathbf{R}| = \mathfrak{c}$.
- Infinite subsets of $^*\mathbf{R}$ of cardinality $< \mathfrak{c}$ are external.
- There are no countably infinite hyper-ordinals.
- Every infinite hyper-integer is uncountable!
- An infinite set is called *hyper-countable* if it has an *internal* 1-1 correspondence with $^*\mathbf{N}$:
  - $^*\mathbf{N}$ and $^*\mathbf{Q}$ are hyper-countable.
  - $^*\mathbf{R}$ is hyper-uncountable.

Dedekind Incompleteness

- Unlike **R**, ***R** is *Dedekind Incomplete* (has "holes" in it)
- Let the set function S : $\mathscr{P}(^*\mathbf{N}) \rightarrow {}^*\mathbf{R}$ be defined as:

$$S(X) = \sum_{n \in X} {}^{1}\!/_{2^n}$$

- We see that: $S(\emptyset) = 0$, $S(\{0,1\}) = 1\frac{1}{2}$, $S(^*\mathbf{N}) = 2$, etc.
- Now suppose $\exists \sigma \in {}^*\mathbf{R} \ni: \ \sigma = S(\mathbf{N})$.
- We have $\sigma > x, \ \forall\, x \in hal(y), \ \forall y \in \mathbf{R} < 2$.
- But $\sigma < x, \ \forall\, x \in hal(2)$.
- Since every hyperreal has a unique shadow: $\sigma \notin {}^*\mathbf{R}$.
- Any attempt to "complete" ***R** by adding elements will simply result in a field isomorphic to **R**.

Proofs from Nonstandard Analysis

- Nonstandard proofs tend to be smaller and more intuitive than their equivalent standard ones
- Standard definition of continuity:

A function f is *continuous* at x_0 if

$$\forall \varepsilon > 0, \exists \delta > 0 \ni: \ |x - x_0| < \delta \ \Rightarrow \ |f(x) - f(x_0)| < \varepsilon.$$

- Nonstandard definition of continuity:

A function f is *continuous* at x_0 if $x \approx x_0 \ \Rightarrow \ f(x) \approx f(x_0)$

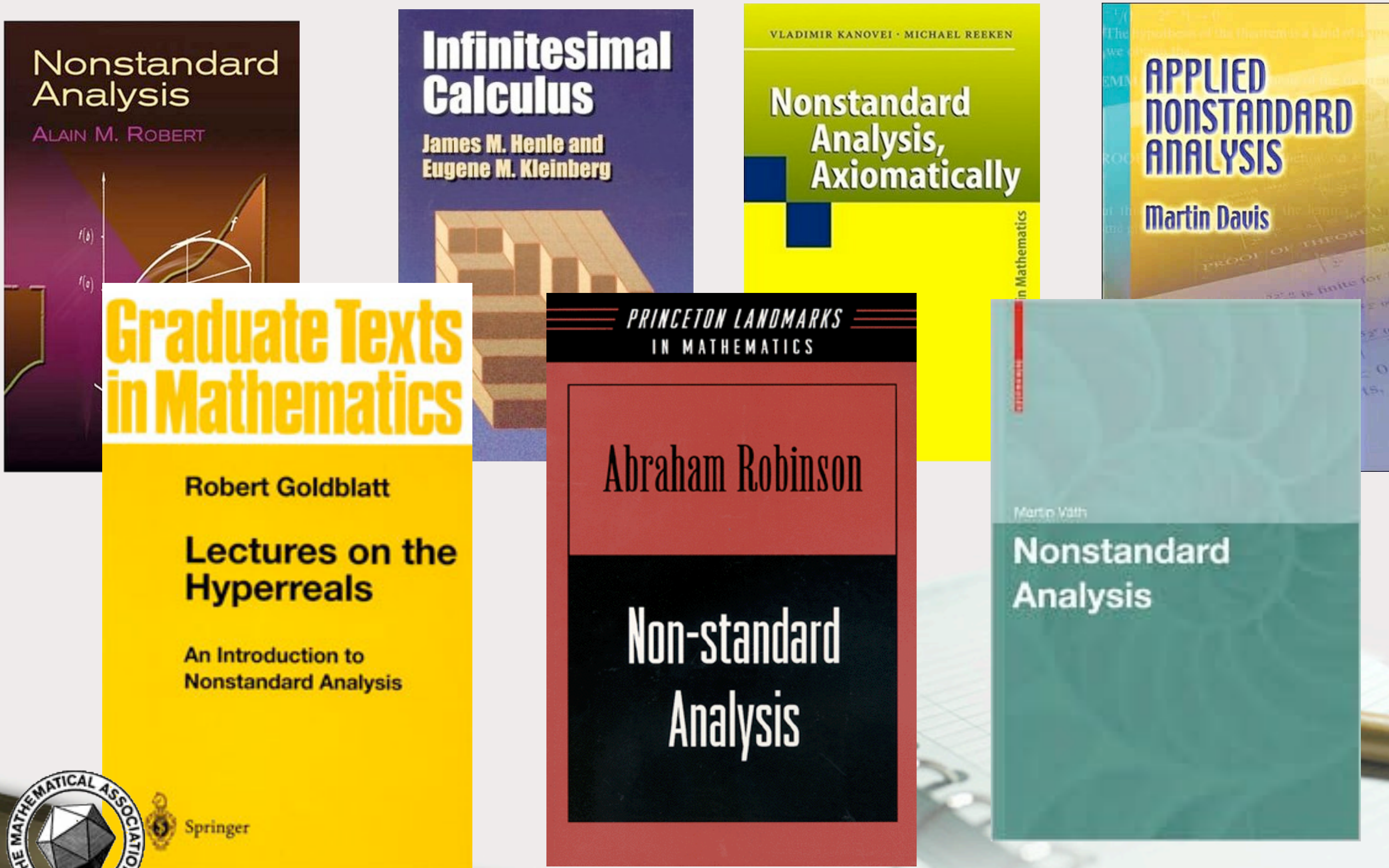
- The paper *Infinitesimals in Modern Mathematics* demonstrates two examples of proofs,

comparing their standard and nonstandard versions.

Conclusion

- ***R** is an extension of **R** containing both infinite and infinitesimal values
- ***R** is a necessarily *incomplete* ordered field
- With the *Transfer Principle*, classical proofs can be rewritten to be more accessible and intuitive.
- Outside the *Transfer Principle*, other new and interesting results from *external sets*.
- Many exciting research opportunities exist for those wishing to learn Nonstandard Analysis

Further Reading...



For more information…

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Q & A

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