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INFINITESIMALS IN MODERN MATHEMATICS

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ABSTRACT

This talk will tour the various modern mathematical models of infinitesimals. After a brief historical overview of their use by mathematicians over the centuries, modern definitions will be introduced. By extending the ordered field of the reals, Non-Archimedean values appear, and depending upon the construction, standard theorems from analysis may produce very unusual results. An introduction of Nonstandard Analysis will be made first, outlining the constriction of Hyperreals and an overview of interesting results pertaining to this structure. This will be followed by a tour of other mathematical systems involving infinitesimals, including Surreal numbers, Superreal numbers, Super-real fields and Smooth Infinitesimal Analysis.

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1. INTRODUCTION & HISTORY

What is an infinitesimal? The mathematical definition we will use here is:

A non-zero number ε is an *infinitesimal* if $|\varepsilon| < 1/n$ for all $n \in \mathbf{N}$.

(Some treatments define infinitesimals to include 0, whilst others do not. For the purposes of this paper, we will assume it does not, although we will allow for exceptions as needed.) It is easily demonstrated that no infinitesimals exist within the field of the real numbers **R**. If ε were an infinitesimal in **R**, then $1/\varepsilon \in \mathbf{R}$ as well. Without loss of generality, assume ε were positive. If we let k be any integer greater than $1/\varepsilon$, then $\varepsilon > 1/k$, which is a contradiction. This simple proof was known to the ancients and would seem to indicate that infinitesimals simply do not exist. Despite this, infinitesimals have had a long and somewhat controversial involvement in mathematics.

The use of infinitesimals can be traced back thousands of years and has been fraught with logical difficulties and paradoxes from the beginning. Archimedes used them for his thought experiments, although he removed them for his final rigorous proofs. Both the infinite and infinitesimal were considered inherently inconsistent concepts at the time and were generally avoided. It was during the development of Calculus that their use became pragmatically necessary. Despite Newton's genius in the discovery of Calculus, he was unable to find a way around postulating their existence. Newton could not adequately defend his inconsistent treatment of infinitesimals (sometimes as zero, sometimes as non-zero), as Bishop Berkeley rightly criticized. It wasn't until 19th century mathematician Karl Weierstrass did Calculus receive a rigorous treatment, removing the references to infinitesimals. At that point, infinitesimals were all but banished from the mathematical world.

In the 1960's, mathematician and logician Abraham Robinson investigated ways in which the concept of the infinitesimal could be revived in a rigorous and consistent manner, avoiding the logical paradoxes which plagued their earlier use. His success in this led to the new branch of mathematics called *Nonstandard Analysis* (NSA). Over the next few decades, other consistent formulations of infinitesimals in number systems were developed, including John Conway's *Surreal Numbers* in the 1970's, David Tall's *Superreal Numbers* in the 1980's, and most recently the *Super-Real Fields* of Dales & Woodin and *Smooth Infinitesimal Analysis* popularized by John L. Bell.

For more on the history of the infinitesimal, visit: http://plato.stanford.edu/entries/continuity/
See also the Wikipedia entry: http://en.wikipedia.org/wiki/Infinitesimal

The purpose of this paper is to tour these various approaches of the modern use of the infinitesimal. A more detailed introduction to Nonstandard Analysis will be given first, as this is by far the most common usage of infinitesimal analysis. This examination will include Hyperreal foundations, results and highlights. Following this, we will tour four other approaches involving infinitesimals. Since many of the concepts of these other approaches are similar with those in NSA, these sections are shorter overviews.

2. THE HYPERREALS (NONSTANDARD ANALYSIS)

In 1965, Abraham Robinson introduced Nonstandard Analysis, a mathematically robust construction of a Non-Archimedean extension of the real numbers. This extension, called the Hyperreals, denoted ***R**, includes both infinitesimal and infinite elements. Since Robinson's original work, there have been further developments of Nonstandard foundations. Today there are essentially three foundational approaches: Robinsonian superstructures, Nelson's axioms to Internal Set Theory, and Keisler's elementary and more intuitive axioms. These three are nicely compared and contrasted by K. D. Stroyan in *The Infinitesimal Rule of Three* published in *Developments in Nonstandard Mathematics* [1995, ISBN: 0582279704]. Each approach has its advantages, and the Stroyan article is recommended for anyone serious about the study of NSA.

In this presentation, I will be constructing ***R** as outlined by Robert Goldblatt's *Lecture on the Hyperreals* [1998, ISBN: 038798464X], as this is both concrete and rigorous.

2.1 $\mathbf{R}^\infty$

First we begin with $\mathbf{R}^\infty$, the set of ordered infinite sequences of **R**. For the purposes of our construction, we will use angle brackets to describe these sequences. Examples of $\mathbf{R}^\infty$ include:

$< 0, 0, 0, 0, ... >$
$< 1, 2, 3, 4, ... >$
$< 1, 0, 1, 0, ... >$
$< 2, 3, 5, 7, ... >$
$< -1, \pi, 0.0001, 10^{10}, \sqrt{17}, ... >$

Every possible sequence of real numbers, convergent or otherwise, are members of $\mathbf{R}^\infty$. We will also use as a convention the subscript notation to refer to specific elements of a sequence. Thus, for $x \in \mathbf{R}^\infty$, we write $x = < x_0, x_1, x_2, ... >$. We will also blur the distinction between **R** and $\mathbf{R}^\infty$ by simply writing the real number r for the sequence $< r, r, r, ...>$. For example, instead of writing $< 3, 3, 3, ... >$, we will simply write the number 3.

Standard arithmetic operations will be extended to $\mathbf{R}^\infty$ in the usual way:

$a + b = < a_0 + b_0, a_1 + b_1, a_2 + b_2, ... >$
$a - b = < a_0 - b_0, a_1 - b_1, a_2 - b_2, ... >$

$a \times b = < a_0 \times b_0, a_1 \times b_1, a_2 \times b_2, \ldots >$
$a \div b = < a_0 \div b_0, a_1 \div b_1, a_2 \div b_2, \ldots >$
$a^b = < a_0^{b_0}, a_1^{b_1}, a_2^{b_2}, a_3^{b_3}, \ldots >$

Where operations are not defined, such as division by 0 or 0^0, we will informally say that the indices of the sequence in which they occur are also undefined. Finally, we will extend functions over **R** to include $\mathbf{R}^\infty$ by the following convention: If $f : \mathbf{R} \to \mathbf{R}$, we define the extension of f as $f : \mathbf{R}^\infty \to \mathbf{R}^\infty$ by:

$f(a) = < f(a_0), f(a_1), f(a_2), \ldots >$

2.2 EQUIVALENCE RELATIONSHIP

At this point in our construction, we have a fully usable $\mathbf{R}^\infty$ structure, but this alone is not sufficient for our definition of ***R**. In particular, we do not have an obvious way of defining a partial ordering. We wish to preserve the natural meaning of < and > under the normal operations of arithmetic.

We would also like to be able to consider two sequences as "equivalent" if they are equal at "almost all" indices. For example, the sequences < 3, 3, 3, 3, ... > and < 3, 0, 3, 3, ... > differ only at the second index and should be considered in all other ways identical.

We wish to define an equivalence relationship ~ on $\mathbf{R}^\infty$ such that $< a_0, a_1, a_2, \ldots > \sim < b_0, b_1, b_2, \ldots >$ is true whenever the set of indices $\mathbf{E} = \{ i \mid a_i = b_i \}$ is considered "large", in some predefined way. The following are the properties we desire for "largeness" and "smallness":

1. All subsets of **N** are either "large" or "small", but not both.
2. All finite sets are "small".
3. All cofinite sets are "large".
4. The complement of a "large" set is "small", and vice-versa.

Furthermore, we need to ensure that our definition of "largeness" supports transitivity of ~. In other words, we wish that whenever a ~ b and b ~ c, that a ~ c always holds. This implies that the intersection of two "large" sets is always "large".

This can be accomplished by using what's called a *non-principal ultrafilter* on **N**. It is not important for the moment to know how a non-principal ultrafilter on **N** is constructed; it suffices to accept only that one exists. (For those interested in understanding more about ultrafilters, visit: http://en.wikipedia.org/wiki/Ultrafilter .)

Now we are in the position to compose our equivalence classes. We write a = b whenever the agreement set $\mathbf{E} = \{ i \mid a_i = b_i \}$ is "large" and $a \neq b$ if **E** is small. Likewise, we can define a < b whenever the agreement set $\{ i \mid a_i < b_i \}$ is large, etc. Moreover, any predicate on real numbers which returns truth or falsity can be extended in this fashion.

The set of equivalence classes we have defined in this quotient ring of $\mathbf{R}^\infty$ is our set ***R**.

Since this definition of ***R** is dependent upon our choice of ultrafilter, there are an infinite number of possible ***R** constructions from which to choose. How different are these various ***R**'s? The answer depends on some set theoretical considerations. If one assumes the Continuum Hypothesis, then the choice of ultrafilter is irrelevant, as all possible ***R**'s are isomorphic. Without the assumption of the Continuum Hypothesis, the situation remains at this time undetermined.

2.3 INFINITIES BOTH GREAT AND SMALL

We are now ready to look at some properties of ***R**. As we have already seen, ***R** contains all of **R**, since each $r \in \mathbf{R}$ is a member of a distinct equivalence class < r, r, r, ... >. What we will now show is that ***R** is a proper superset of **R**. Let us define:

$\omega = < 1, 2, 3, 4, \ldots >$

To see that ω is infinite, note that for any natural number n, the agreement set $\{ i \mid i > n \}$ is cofinite, and thus $\omega > n$ for all n. Likewise, we can consider the inverse of ω, called ε:

$\varepsilon = 1/\omega = < 1, 1/2, 1/3, 1/4, \ldots >$

By similar inspection, we see that ε is smaller than any positive real number and thus is infinitesimal. With exponentiation operations defined in ***R**, there are examples of increasingly small hyperreals (ε^2, ε^{100}, ε^{ω}, etc.) as well as ones which are increasingly large (ω^2, ω^{ω}, $\omega^{\omega^{\omega}}$, etc.). We can even calculate combined infinite and infinitesimal arithmetic, such as:

ω^{ε} = <1.000..., 1.414..., 1.442..., 1.414...., 1.379... >

2.4 HYPERREAL TERMINOLOGY & RESULTS

Below are some interesting results generated from NSA:

1. We say x is *infinitely close* to y, if $x - y$ is infinitesimal, written $x \approx y$. (For example, $\omega^{\varepsilon} \approx 1$).
2. The set of hyperreals infinitely close to x is called the *halo* of x, denoted *hal(x)*.
3. The set of infinitesimals is simply *hal(0)*.
4. If $x \in$ ***R**, $r \in$ **R** and $x \approx r$, then we call r the *shadow* of x, denoted $r = shd(x)$.
5. The set of all hyperreals a finite distance from x is called the *galaxy* of x, denoted *gal(x)*.
6. The set of finite hyperreals is simply *gal(0)*.
7. Every finite hyperreal is infinitely close to **exactly one** real number.
8. For $r \in$ **R**, $x \in$ ***R**, $x \approx r$, we call r the *standard part* of x.
9. For $r \in$ **R**, $x \in$ ***R**, $x \approx r$, we call $x - r$ the *nonstandard part* of x.
10. All finite hyperreals can be uniquely expressed as $r + i$ where r is a real number and i is an infinitesimal.

2.5 INTERNAL SETS & THE TRANSFER PRINCIPLE

Sets within Nonstandard Analysis are divided into two types: *internal* sets and *external* sets. Internal sets are those which have analogs in standard analysis over the *-transformation. For example, the *-transformation maps the unit interval U = [0,1] in **R** to the internal set *U = [0,1] in ***R**. Although U and *U may appear identical at first glance, they are quite different sets. For one thing, *U contains infinitesimals whereas U does not. The set U of standard reals also exists in ***R**, but it is an external set not an internal one.

The Transfer Principle says that statements and properties true of standard analysis have "equivalent" statements and properties true in Nonstandard Analysis. These equivalents, however, involve only internal elements of NSA via the *-transformation. A concrete example may help elucidate this point. Consider the Archimedian Property of real numbers:

For any x, there exists $n \in$ **N** such that $n > x$.

In **R**, this property true, but in ***R** it is false as currently stated. To get the true equivalent in NSA, we must replace the sets described within the statement into their *-transformations. Doing so produces the following true statement of ***R**:

For any x, there exists $n \in$ ***N** such that $n > x$.

2.6 NON-STANDARD PROOFS

If the Transfer Principle implies that NSA contains all of the same theorems of classical analysis, what is the benefit gained from studying NSA? Although it is true that the theorems of standard analysis hold true in NSA (under the *-transformation), it is untrue that only *- transferred theorems are theorems of NSA. Theorems involving external sets are one without classical analogs. One such theorem is: the cardinality of ***N** is equal to the cardinality of **R** (the size of the continuum). Such results are not demonstrable from the Transfer Principle alone.

More importantly, however, are the pedagogical benefits of using Nonstandard proofs. Nonstandard proofs are typically more intuitive than standard proofs. Consider, for example, the standard definition of continuity:

A function f is *continuous* at a point x_0 if: for every $\varepsilon > 0$, there exists $\delta > 0$ such that $|x - x_0| < \delta \Rightarrow |f(x) - f(x_0)| < \varepsilon$.

Such a definition obfuscates the intended conceptual meaning of continuity. Students must memorize a mathematical formulation, rather than understand the concept. Worse still, the standard "delta-epsilon" proofs are often exceedingly difficult for novice to follow.

Consider instead the nonstandard definition of continuity:

A function f is *continuous* at a point x_0 if: when $x \approx x_0$ then $f(x) \approx f(x_0)$.

Memorization is less important, as they key concept is made obvious: continuity essentially means that infinitely close points map to infinitely close points. For f to be continuous at x_0, it is sufficient to show that $f(x_0 + i) \approx f(x_0)$ for arbitrary infinitesimal i.

Nonstandard proofs tend to be much shorter as well. Let us compare two proofs for the continuity of the function $f(x) = ax + b$. The first will be a standard delta-epsilon proof, and the second will be a nonstandard proof.

Standard Proof: Let x_0 be arbitrary. We note that

$|f(x) - f(x_0)|$
$= |ax + b - ax_0 - b|$
$= |ax - ax_0|$
$= |a| \cdot |x - x_0|$
Therefore, by choosing $\delta = \varepsilon / |a|$, we have:
$|x - x_0| < \delta$
$\Rightarrow |x - x_0| < \varepsilon / |a|$
$\Rightarrow |a| \cdot |x - x_0| < \varepsilon$
$\Rightarrow |ax - ax_0| < \varepsilon$
$\Rightarrow |ax + b - ax_0 - b| < \varepsilon$
$\Rightarrow |f(x) - f(x_0)| < \varepsilon$
Since x_0 was arbitrary, we have this holding for all x.
Q. E. D.

Nonstandard Proof: Let i be an arbitrary infinitesimal. We note that

$f(x + i)$
$= a(x + i) + b$
$= (ax + b) + ai$
$= f(x) + ai$
$\approx f(x)$
Q. E. D.

2.7 HYPERREAL INCOMPLETENESS

It is natural to suppose that by enlarging the reals to include additional elements, we have somehow made the number line more complete. It then may come as a surprise to you that the opposite is the case: ***R** is, in fact, *less* complete than **R**.

To understand this better, it is worth looking at another well known incomplete set: **Q**, the set of rationals. Consider the equation $f(x) = x^2 - 2$. For no $q \in$ **Q** do we have $f(q) = 0$. One can choose rationals that can get f as close to 0 as we like, but no rational number will get to 0 exactly. In other words, **Q** has a "hole" in it where √2 should be.

Dedekind formalized the concept of completeness in the following manner: A partially ordered set is *complete* if every subset with an upper bound has a least upper bound. Take for example, the set T = { 1, 1.4, 1.41, 1.414, 1.4142, ... }. It is bounded above, but T has no least upper bound in **Q**. However, T has a least upper bound in **R**, namely √2. This is because **R** is a complete ordered field, whereas **Q** is an incomplete ordered field.

It is easy to prove that ***R** is incomplete as well:

Theorem: ***R** is Dedekind Incomplete.

Proof: Let **I** = the set of infinitesimals. Note that any positive real number is an upper bound of **I**. Assume **I** has a least upper bound δ. Either $\delta \in$ **I** or $\delta \notin$ **I**. If $\delta \in$ **I**, then $2\delta \in$ **I**. But $2\delta > \delta$, which contradicts the assumption that δ is a least upper bound. So assume $\delta \notin$ **I**. Then $\delta/2 \notin$ **I**, and so $\delta/2$ is an upper bound of **I** as well. But since $\delta/2 < \delta$, δ cannot be the least upper bound. Therefore, I has no least upper bound, and hence ***R** is incomplete.

So then a question that might arise is this: Can one "complete" ***R** by continually appending missing elements to it? Unfortunately, this has an unsatisfying answer. It has been proven that every complete ordered field is isomorphic to **R**. Thus, to "complete" ***R**, the elements needing to be added are precisely those which remove the gaps and distinctions between the infinitesimals, the finite and the infinite. You essentially get back **R** (or something indistinguishable from it).

Although incomplete, ***R** does however have the property of being *internally complete*: that is, all *internal* subsets of ***R** with an upper bound have a least upper bound. This is due to the Transfer Principle. The converse of this can be used to prove the externality of some sets. For example: **I**, **N**, and **R** each has an upper bound but no least upper bound; therefore, each of these sets must be external.

Interestingly, hyperreal incompleteness implies that there are equations in ***R** which have solutions for internal sets but not for external ones. For example, consider the set function S(X) as:

$$S(X) = \sum_{n \in X} \tfrac{1}{2^n} \text{ , for X a subset of } {}^*\mathbf{N}.$$

S(X) has a solution for any internal X, eg: *S(*N)* = 2. However, *S(X)* does not have a solution for every external set. X = **N** has no solution, as *S(N)* exceeds any value in *hal(x)* for $x < 2$, but is always strictly less than any value in *hal(2)*. *S(N)* essentially "points" to a hole in ***R**.

3. OTHER APPROACHES

Discussions involving the modern use of infinitesimals usually refer to those in the hyperreal system of Nonstandard Analysis. However, the past few decades of mathematical research have developed alternative approaches to extending **R** which include non-Archimedean values. We will tour four of them: Conway's *Surreal* numbers, Tall's *Superreal* numbers, Dales & Woodin's *Super-real* numbers and finally Bell's *Smooth Infinitesimal Analysis*.

3.1 SURREAL NUMBERS

John Conway's construction of surreal numbers is reminiscent of Dedekind cuts used to generate real numbers. The surreal construction is recursive cut of two sets of other surreal numbers.

> Construction Rule: If *L* and *R* are two sets of surreal numbers, such that no member of *R* is less than or equal to any member of *L*, then the ordered pair denoted { *L* | *R* } is a surreal number.

Essentially, a surreal number is two sets of surreals with the left side *L* being smaller than the right side *R*. We also allow *L* and/or *R* to be empty. In the case where *L* is the empty set, we simply write { | *R* } for { ∅ | *R* }; likewise { *L* | } for { *L* | ∅ } and { | } for { ∅ | ∅ }. We will define the surreal number 0 as { | }. To define "larger" and "smaller", we need an inequality definition:

> Comparison Rule: For two surreal numbers a and b, where a = { a_L, a_R }, b = { b_L, b_R }, we say that a ≤ b if and only if b is less than or equal to no member of a_L and no member of b_R is less than or equal to a.

Essentially, all the elements in *L* need to be greater than the elements in *R* for { *L* | *R* } to be a valid surreal number (allowing for the empty set in either *L* or *R*. Equality is defined, as you would expect as follows: a = b if a ≤ b and b ≤ a.

As stated above, we begin by postulating our first surreal number 0 = { | }. With 0 in place, we can now apply the first recursive iteration, generating these three potential numbers: { 0 | }, { | 0 } and { 0 | 0 }. This final one is not a valid surreal number since 0 ≤ 0, so we are left with the first two newly generated surreal numbers, which we name 1 and -1 respectively.

For the second iteration, all combinations of 0, 1 -1 and the empty set are examined. Many of them, such at { 0 | 1, -1 }, are not valid surreal numbers, since they fail the comparison rule. Others turn out to be equal to equivalent classes we already have, such as { -1 | 1 } = { | } = 0. However, four new equivalence classes are generated: { 1 | }, { | -1 }, { 0 | 1 } and { -1 | 0 }. These we name 2, -2, ½, -½ respectively.

The third iteration produces the surreal numbers 3, -3, ¼, ¾, -¼ and -¾. In general, the nth generation will produce ±n, and all multiples of $\pm\frac{1}{2^n}$. We define a surreal number's *birthday* as the index number corresponding to the iteration in which it is generated. For example, the birthday of -3 is 3, and the birthday of 9⁄16 is 5.

We define S_i as the set of all surreal numbers whose birthday is ≤ i. We define S_ω as the union of all S_i for i ∈ **N**. Do we have all the numbers we want in S_ω? Unfortunately, most of the members of **R** remaining missing, even at S_ω. This is because at no finite iteration do rational numbers like ⅓ appear. The solution is to extend this process through transfinite induction, yielding the missing numbers. In general, we allow birthdays of any transfinite ordinal number.

Our very next birthday, ω+1, turns out to be very interesting. As you might expect, the remainder of our rational numbers come in at this point. What is surprising is that ω+1 is also the birthday of all the remaining reals as well! It is a bit unexpected that transcendental numbers like π share the same birthday as a pedestrian rational such as $\frac{1}{3}$. It doesn't stop there, as $\omega+1$ is the birthday of our first infinitesimal! The surreal number defined as

$$\varepsilon = \{ 0 \mid \ldots \tfrac{1}{8}, \tfrac{1}{4}, \tfrac{1}{2}, 1 \}$$

comes in iteration $\omega+1$ and is provably an infinitesimal, since $\varepsilon < \frac{1}{n}$ for all finite n. Its inverse is also generated at this point:

$$\omega = \{ S_\omega \mid \} = \{ 1, 2, 3, \ldots \mid \}$$

With additional iterations, more and more infinitesimals and infinite numbers become generated. It turns out that the entirety of **On** (the class of ordinals) are eventually generated and become members of the class of surreals. Moreover, new and unusual numbers never defined before begin appearing in later transfinite iterations, such as strange beasts as:

$$\sqrt{\omega} = \{ 1, 2, 3, \ldots \mid \ldots \tfrac{\omega}{8}, \tfrac{\omega}{4}, \tfrac{\omega}{2}, \omega \}$$
$$\log \omega = \{ 1, 2, 3, \ldots \mid \ldots \sqrt[4]{\omega}, \sqrt[3]{\omega}, \sqrt{\omega}, \omega \}$$

It has been speculated that the surreal numbers encompass the largest class of numbers possible.

3.2 SUPERREAL NUMBERS [Tall]

A less ambitious but much more accessible approach to defining infinitesimals is one by David Tall from the University of Warwick. His motivation was to create a system which was more intuitive for students and to make Calculus concepts easier to grasp. The simplicity of his approach is very appealing, as it quickly gets to the use of infinitesimals without the large construction found ***R**'s construction.

Tall begins with **R** and appends a postulated infinitesimal ε. Other infinitesimals are generated by closure over the basic arithmetic operations of +, -, ×, ÷, and the result is the set of superreals $\mathscr{R}$ (not to be confused with the super-reals of Dales & Woodin, described below). Every superreal can be uniquely expressed as:

$$z = a_n\varepsilon^n + a_{n-1}\varepsilon^{n-1} + \ldots + a_1\varepsilon + a_0 + a_{-1}\varepsilon^{-1} + \ldots + a_{-m}\varepsilon^{-m} \quad \text{for } a_i \in \mathbf{R},\ n,m \in \mathbf{N}$$

Note that whenever $a_i = 0$ for all non-zero i, z is simply a real number. Infinitesimals are of the form:

$$\delta = a_n\varepsilon^n + a_{n-1}\varepsilon^{n-1} + \ldots + a_1\varepsilon$$

By allowing ε to have an inverse, the superreals also contain infinite values as well. As this is less useful for Calculus purposes, one can also construct $\mathscr{R}$ such that ε is not invertible, and thus all superreals would be finite but still contain infinitesimals.

Tall's construction entails a pragmatic approach to function extension. Whereas any real function can be extended into the hyperreals, only analytic functions are considered for superreal extension. Since analytic functions are expressible as power series, superreal functions can be defined by extending the series into the infinitesimals. For example:

$$\sin \delta = \delta - \delta^3/3! + \delta^5/5! + \ldots$$
$$e^\delta = 1 + \delta + \delta^2/2! + \delta^3/3! + \ldots$$

Tall's superreals are algebraic in nature and do not carry the baggage of complicated superstructures or the need for First Order Logic. $\mathscr{R}$ is not as all-encompassing as ***R**, but it does pragmatically provide most of the intuitive benefits of NSA without the same cognitive overhead. The result is an Infinitesimal Calculus which is reminiscent of Leibniz.

For more information on David Tall and his theory of superreals, visit: http://www.warwick.ac.uk/staff/David.Tall/themes/limits-infinity.html

3.3 SUPER-REAL NUMBERS [Dales & Woodin]

Another extension of real numbers calling itself *superreal* was introduced by Garth Dales & Hugh Woodin in their 1996 text *Super-Real Fields: Totally Ordered Fields with Additional Structure* [ISBN: 0198536437]. The *super-reals* described therein should not be confused with those of the same name by David Tall described above. Conveniently, the two can be distinguished by observing the convention that Dales & Woodin hyphenate the term whereas Tall does not.

The super-real field is a more abstract extension of **R**, in fact more abstract than even ***R**. It begins with the ring C(X) of real-valued continuous functions on a topological space X. X can be any Tychonoff space, not necessarily **R**. If P is a prime ideal of C(X), and A is a factor algebra of C(X) / P, we let *F* be a quotient field of A strictly containing **R**. If *F* is not order-isomorphic to **R**, then *F* is a super-real field. When P is the maximal ideal, it is shown that F is the hyperreal field, showing that super-real fields are more general than the hyperreal ones.

The material presented is one of models and does not delve into the particulars of Calculus. It presumably shares a great deal in common with ***R**, since superreals are essentially generalizations of hyperreal fields.

For more information on super-real fields by Dales & Woodin, visit the American Mathematical Society review: http://www.ams.org/bull/1998-35-01/S0273-0979-98-00740-X/S0273-0979-98-00740-X.pdf

3.4 SMOOTH INFINITESIMAL ANALYSIS

John L. Bell describes an unusual approach to the topic with Smooth Infinitesimal Analysis (SIA). There is no predefined construction involving models within **R**, as there is with the hyperreals or surreals. Instead, he introduces the concept of the *nilpotent infinitesimal*, that is, a nonzero number so small that its square is 0. The set of infinitesimals is simply defined as:

$$\mathbf{I} = \{ x \mid x^2 = 0 \}$$

In standard analysis, $\mathbf{I} = \{ 0 \}$; in SIA, there exists nonzero elements of **I**. These smooth infinitesimals are not invertible, and so do not form a field. Thus, there are no infinite values in SIA. The non-invertibility of smooth infinitesimals prevents general division by them, but this is compensated for by the Infinitesimal Cancellation Law, which says:

For all $x,y \in \mathbf{R}$, if $x \cdot \varepsilon = y \cdot \varepsilon$ then $x = y$.

Although unusual, this system has some interesting properties from the perspective of Calculus. For example, for a function *f* and infinitesimal ε, we have:

$$f(x + \varepsilon) = f(x) + f'(x) \cdot \varepsilon$$

Geometrically, this implies that functions are linear over an infinitesimal interval. Curves are essentially the concatenation of infinitesimally straight lines. This is called the property of *microstraightness*. A surprising result from microstraightness is the proof that *all* functions are continuous and infinitely differentiable! This is why this branch of analysis is called *Smooth*.

There is no Transfer Principle in SIA, so many standard theorems of classical analysis do not hold in this approach (for example, the Intermediate Value Theorem is false). The single major drawback to SIA is its inconsistency with the *Law of the Excluded Middle*. This deeply undermines the conceptual gains made by SIA, leaving a great deal of cognitive friction. Despite this deficiency, Smooth Infinitesimal Analysis has generated a fascinating new perspective on the old subject of Calculus.

For more information on Smooth Infinitesimal Analysis, visit: http://publish.uwo.ca/~jbell/invitation%20to%20SIA.pdf .

4. CONCLUSION

Abraham Robinson introduced a rigorous system of infinitesimals into modern mathematics, avoiding the pitfalls, contradictions and paradoxes which plagued their use in centuries past. In the four decades since, much more investigation and study has been accomplished with greater understanding of Nonstandard Analysis. New alternative number systems involving infinitesimals are already in use, including Conway's surreal numbers and Bell's nilpotent infinitesimals. The 21st century surely holds new and exciting developments in this area.

5. FURTHER READING

Nonstandard Analysis:
- Goldblatt, *Lecture on the Hyperreals* [1998, ISBN: 038798464X]
- Robinson, *Non-standard Analysis* [1996, ISBN: 0691044902]
- Davis, Applied *Nonstandard Analysis* [1977, 2005, ISBN: 0486442292]
- Cutland et al, *Developments in Nonstandard Mathematics* [1995, ISBN: 0582279704]
- Vath, *Nonstandard Analysis* [2007, ISBN: 3764377739]
- Stroyan & Luxemburg, *Introduction to the Theory of Infinitesimals* [1976, ISBN: 0126741506]

Hermann, *Nonstandard Analysis: A Simplified Approach* [2003, http://arxiv.org/pdf/math.GM/0310351.pdf]
Kim, *Nonstandard Analysis and Applications* [2006, http://www.math.ucdavis.edu/~ekim/talks/nonstandard_analysis.pdf]
Lutz & Goze, *Nonstandard Analysis* [1981, ISBN: 0387108793]
Diener & Diener, *Nonstandard Analysis in Practice* [1991, ISBN: 3540602976]
Hurd & Loeb, *An Introduction to Nonstandard Real Analysis* [1985, ISBN: 0123824401]
Wikipedia entry: http://en.wikipedia.org/wiki/Hyperreal_number
Wikipedia entry: http://en.wikipedia.org/wiki/Non-standard_analysis

Surreal Numbers:
Conway, *On Numbers and Games* [1976, ISBN: 1568811276]
Alling, *Foundations of Analysis Over Surreal Number Fields* [1987, ISBN: 0444702261]
Gonshor, *An Introduction to the Theory of Surreal Numbers* [1986, ISBN: 0521312051]
Wikipedia entry: http://en.wikipedia.org/wiki/Surreal_number

Superreal Numbers [Tall]:
Tall, *Intuitive Infinitesimals in Calculus* [http://www.warwick.ac.uk/staff/David.Tall/pdfs/dot1980c-intuitive-infls.pdf]
Tall, *Mathematical Intuition* [http://www.warwick.ac.uk/staff/David.Tall/pdfs/dot1980d-int-lims-pme.pdf]
Tall, *The Notion of Infinite Measuring Number* [http://www.warwick.ac.uk/staff/David.Tall/pdfs/dot1980b-inf-measuring-num.pdf]

Super-Real Numbers [Dales & Woodin]:
Dales & Woodin, *Super-Real Fields: Totally Ordered Fields with Additional Structure* [1996, ISBN: 0198539916]
Wikipedia entry: http://en.wikipedia.org/wiki/Superreal

Smooth Infinitesimal Analysis:
Bell, *A Primer of Infinitesimal Analysis* [1998, ISBN: 0521624010]
Wikipedia entry: http://en.wikipedia.org/wiki/Smooth_infinitesimal_analysis